

A Support-Reduction Principle for Variational Trace Functionals

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We establish a simultaneous support-reduction principle for optimization problems involving a class of variational trace functionals. Applications include three-parameter conditional Rényi entropies, optimized α - z conditional entropies, and α - z extensions of doubly minimized Rényi mutual information, Augustin information, and Rényi information radii.

1. INTRODUCTION

Numerous definitions of quantum information quantities often involve optimization over an auxiliary density operator. When the fixed states are not full rank, the optimization domain may be larger than the support relevant to the quantity under consideration. A natural question is whether the auxiliary state can be restricted to this support without changing the optimal value. Such a reduction lowers the dimension of the optimization problem and simplifies the treatment of operator powers and support conditions.

This question is particularly useful when the optimization variable appears with a negative exponent. Negative powers of singular operators are commonly defined through generalized inverses [1, 2], using the Moore–Penrose pseudoinverse on the original Hilbert space. These inverse powers need not be continuous when the rank changes. If the optimization can instead be restricted to states whose support equals a fixed subspace, then the auxiliary states are positive definite on that subspace. Their negative powers can consequently be treated as ordinary inverse powers, which are continuous on the positive-definite domain of the reduced space.

In this note, we establish a support-reduction principle for the following class of variational trace functionals.

Definition 1.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be finite-dimensional Hilbert spaces, let $P \in \text{Pos}(\mathcal{X})$ be a projection, and let $\Phi \in \text{T}(\mathcal{X}, \mathcal{Y})$ be a positive map satisfying $\Phi(X) = \Phi(PXP)$ for every $X \in \text{Pos}(\mathcal{X})$. For $r \in \mathbb{R}$ and $z \in (0, \infty)$, define the trace functional*

$$F_{r,z}(\sigma) := \text{Tr}(\Phi(\sigma^r))^z, \quad \sigma \in \text{Pos}(\mathcal{X}). \quad (1)$$

For nonzero r , the compression invariance on Φ ensures that the power compression $\hat{\sigma} = (P\sigma^r P)^{1/r}$ leaves $F_{r,z}$ unchanged. For the admissible states in Proposition 2.4, spectral interlacing gives $0 < \text{Tr} \hat{\sigma} \leq 1$, allowing $\hat{\sigma}$ to be normalized to a density operator. The same normalized state yields simultaneous comparisons for a finite family of positive maps satisfying $\Phi_i(X) = \Phi_i(PXP)$. Theorem 2.5 converts these comparisons into support reductions for positively weighted logarithmic sums and the corresponding minimum or maximum objectives, treating positive, zero, and negative powers separately.

Our starting motivation is the optimization in the three-parameter family of conditional Rényi entropies [2]. We also apply the principle to optimized α - z conditional entropies [2], doubly minimized Rényi mutual information [3], Augustin information [4], and Rényi divergence radii [5]. These support reductions do not require a data-processing inequality for the underlying entropy or divergence, and thus impose no parameter restrictions associated with such an inequality. These examples illustrate single-objective optimization, successive reductions of multiple reference states, and simultaneous reductions for weighted and extremal objectives. The next section establishes the general principle and its single-map corollary; the final section gives the applications.

Throughout, all Hilbert spaces are finite dimensional. We write $\text{Herm}(\mathcal{H})$ and $\text{Pos}(\mathcal{H})$ for the Hermitian and positive semidefinite operators on $\mathcal{H}$, respectively, and $\text{S}(\mathcal{H})$ for the positive

semidefinite operators of trace one. The notation $T(\mathcal{X}, \mathcal{Y})$ denotes the space of linear maps between the corresponding operator spaces. For $X \geq 0$, $\text{supp}(X)$ is its range. For $r > 0$, X^r is the usual power; for $r = 0$, $X^0 = \Pi_X$, the orthogonal projection onto $\text{supp}(X)$; and for $r < 0$, negative powers are defined using the Moore–Penrose pseudoinverse. We write $P \perp \sigma$ when their supports are orthogonal. All logarithms use the base two, with $\log 0 = -\infty$.

2. THE SUPPORT-REDUCTION PRINCIPLE

We begin with the trace functional in Definition 1.1 and its compression identity $\Phi(X) = \Phi(PXP)$. Direct compression to $P\sigma P$ generally does not preserve $F_{r,z}$, since $(P\sigma P)^r$ need not equal $P\sigma^r P$. For $r \neq 0$, the compression identity instead allows us to construct an operator supported on $\text{supp}(P)$ with the same value of $F_{r,z}$. The construction preserves the functional but need not preserve the trace.

Lemma 2.1. *For $r \neq 0$, every $\sigma \in \text{Pos}(\mathcal{X})$ admits $\hat{\sigma} = (P\sigma^r P)^{1/r}$ satisfying $\text{supp}(\hat{\sigma}) \subseteq \text{supp}(P)$ and*

$$F_{r,z}(\hat{\sigma}) = F_{r,z}(\sigma). \quad (2)$$

Proof. Trivial. $\square$

For a Hermitian operator X , we denote the eigenvalues of X in non-increasing order by

$$\mu_1(X) \geq \mu_2(X) \geq \cdots \geq \mu_{\dim(\mathcal{H})}(X). \quad (3)$$

Throughout the remainder of this section, all eigenvalue comparisons are understood with respect to this ordering convention.

Lemma 2.2. *Let $\tau \in \text{Pos}(\mathcal{X})$ be full rank. If $r = 0$, then*

$$\sup_{\sigma \in \text{Pos}(\mathcal{X})} F_{0,z}(\sigma) = F_{0,z}(\tau). \quad (4)$$

Proof. For every $\sigma \in \text{Pos}(\mathcal{X})$, we have $\sigma^0 = \Pi_\sigma \leq I_{\mathcal{X}} = \tau^0$. Hence, $\Phi(\Pi_\sigma) \leq \Phi(I_{\mathcal{X}})$. Let $d = \dim(\mathcal{Y})$. By the min–max theorem, $\mu_k(\Phi(\Pi_\sigma)) \leq \mu_k(\Phi(I_{\mathcal{X}}))$ for every $k = 1, \dots, d$. Since $z > 0$, it follows that

$$F_{0,z}(\sigma) = \text{Tr}(\Phi(\Pi_\sigma))^z = \sum_{k=1}^d \mu_k(\Phi(\Pi_\sigma))^z \leq \sum_{k=1}^d \mu_k(\Phi(I_{\mathcal{X}}))^z = \text{Tr}(\Phi(I_{\mathcal{X}}))^z = F_{0,z}(\tau). \quad (5)$$

Therefore,

$$\sup_{\sigma \in \text{Pos}(\mathcal{X})} F_{0,z}(\sigma) = F_{0,z}(\tau), \quad (6)$$

as claimed. $\square$

Lemma 2.3 (Cauchy interlacing theorem). *Let $\mathcal{H}$ be an n -dimensional Hilbert space, $X \in \text{Herm}(\mathcal{H})$, and $P \in \text{Pos}(\mathcal{H})$ be an orthogonal projection of rank k . Regarding PXP as a Hermitian operator on $\text{ran}(P)$, we have*

$$\mu_i(X) \geq \mu_i(PXP) \geq \mu_{n-k+i}(X), \quad i = 1, \dots, k. \quad (7)$$

Proof. The result follows directly from the Min-Max theorem. $\square$

Lemma 2.1 gives an operator with the desired support, but optimization over density operators also requires trace one. We therefore need to analyze its trace before normalizing.

Proposition 2.4 (Compressed normalization). *Let $P \in \text{Pos}(\mathcal{X})$ be a nonzero orthogonal projection, let $\sigma \in \text{S}(\mathcal{X})$, and let $r \in \mathbb{R} \setminus \{0\}$. Assume $P \not\perp \sigma$ when $r > 0$, and $\text{supp}(P) \subseteq \text{supp}(\sigma)$ when $r < 0$. Then*

$$0 < \text{Tr}(P\sigma^r P)^{1/r} \leq 1. \quad (8)$$

Proof. Set $k = \text{rank}(P)$ and $c = \text{Tr}(P\sigma^r P)^{1/r}$. In both cases, $c > 0$.

Suppose that $r > 0$. By Lemma 2.3,

$$\mu_i(P\sigma^r P) \leq \mu_i(\sigma^r) = \mu_i(\sigma)^r, \quad i = 1, \dots, k. \quad (9)$$

Since $1/r > 0$, we obtain

$$c = \sum_{i=1}^k \mu_i(P\sigma^r P)^{1/r} \leq \sum_{i=1}^k \mu_i(\sigma) \leq \text{Tr}(\sigma) = 1. \quad (10)$$

Now suppose that $r < 0$ and $\text{supp}(P) \subseteq \text{supp}(\sigma)$. We work on $\text{supp}(\sigma)$, on which σ is positive definite. Let $m = \dim \text{supp}(\sigma)$. Thus, we have

$$\mu_1(\sigma) \geq \mu_2(\sigma) \geq \dots \geq \mu_m(\sigma) > 0, \quad (11)$$

$$\mu_m(\sigma)^r \geq \mu_{m-1}(\sigma)^r \geq \dots \geq \mu_1(\sigma)^r > 0. \quad (12)$$

The eigenvalues of σ^r occur in the reverse order. Lemma 2.3 therefore gives

$$\mu_i(P\sigma^r P) \geq \mu_{m-k+i}(\sigma^r) = \mu_{k-i+1}(\sigma)^r, \quad i = 1, \dots, k. \quad (13)$$

As $1/r < 0$, taking the power $1/r$ reverses the inequality and hence

$$c = \sum_{i=1}^k \mu_i(P\sigma^r P)^{1/r} \leq \sum_{i=1}^k \mu_{k-i+1}(\sigma) = \sum_{i=1}^k \mu_i(\sigma) \leq 1. \quad (14)$$

Thus $0 < c \leq 1$ as desired. $\square$

We are now ready to prove the simultaneous support-reduction principle.

Theorem 2.5 (Simultaneous support reduction). *Let $\mathcal{X}$ and $\mathcal{Y}$ be finite-dimensional Hilbert spaces, let $P \in \text{Pos}(\mathcal{X})$ be a nonzero orthogonal projection, and let $\{\Phi_i\}_{i \in \mathcal{I}}$ be a nonempty finite family of positive maps in $\text{T}(\mathcal{X}, \mathcal{Y})$ satisfying $\Phi_i(X) = \Phi_i(PXP)$ for every $X \in \text{Pos}(\mathcal{X})$ and $i \in \mathcal{I}$. For $r \in \mathbb{R}$ and $z > 0$, write*

$$F_{r,z}^{(i)}(\sigma) = \text{Tr}(\Phi_i(\sigma^r))^z, \quad i \in \mathcal{I}. \quad (15)$$

Let $p_i > 0$ for every $i \in \mathcal{I}$. For $r > 0$, the following support reductions hold:

$$\sup_{\sigma \in \text{S}(\mathcal{X})} \sum_{i \in \mathcal{I}} p_i \log F_{r,z}^{(i)}(\sigma) = \sup_{\substack{\sigma \in \text{S}(\mathcal{X}) \\ \text{supp}(\sigma) \subseteq \text{supp}(P)}} \sum_{i \in \mathcal{I}} p_i \log F_{r,z}^{(i)}(\sigma), \quad (16)$$

$$\sup_{\sigma \in \text{S}(\mathcal{X})} \min_{i \in \mathcal{I}} \log F_{r,z}^{(i)}(\sigma) = \sup_{\substack{\sigma \in \text{S}(\mathcal{X}) \\ \text{supp}(\sigma) \subseteq \text{supp}(P)}} \min_{i \in \mathcal{I}} \log F_{r,z}^{(i)}(\sigma). \quad (17)$$

For $r = 0$, we have

$$\sup_{\sigma \in \mathcal{S}(\mathcal{X})} \sum_{i \in \mathcal{I}} p_i \log F_{0,z}^{(i)}(\sigma) = \sup_{\substack{\sigma \in \mathcal{S}(\mathcal{X}) \\ \text{supp}(\sigma) = \text{supp}(P)}} \sum_{i \in \mathcal{I}} p_i \log F_{0,z}^{(i)}(\sigma), \quad (18)$$

$$\sup_{\sigma \in \mathcal{S}(\mathcal{X})} \min_{i \in \mathcal{I}} \log F_{0,z}^{(i)}(\sigma) = \sup_{\substack{\sigma \in \mathcal{S}(\mathcal{X}) \\ \text{supp}(\sigma) = \text{supp}(P)}} \min_{i \in \mathcal{I}} \log F_{0,z}^{(i)}(\sigma), \quad (19)$$

$$\sup_{\sigma \in \mathcal{S}(\mathcal{X})} \max_{i \in \mathcal{I}} \log F_{0,z}^{(i)}(\sigma) = \sup_{\substack{\sigma \in \mathcal{S}(\mathcal{X}) \\ \text{supp}(\sigma) = \text{supp}(P)}} \max_{i \in \mathcal{I}} \log F_{0,z}^{(i)}(\sigma). \quad (20)$$

All three suprema are attained by $P/\text{Tr}(P)$. For $r < 0$, we have

$$\inf_{\substack{\sigma \in \mathcal{S}(\mathcal{X}) \\ \text{supp}(P) \subseteq \text{supp}(\sigma)}} \sum_{i \in \mathcal{I}} p_i \log F_{r,z}^{(i)}(\sigma) = \inf_{\substack{\sigma \in \mathcal{S}(\mathcal{X}) \\ \text{supp}(\sigma) = \text{supp}(P)}} \sum_{i \in \mathcal{I}} p_i \log F_{r,z}^{(i)}(\sigma), \quad (21)$$

$$\inf_{\substack{\sigma \in \mathcal{S}(\mathcal{X}) \\ \text{supp}(P) \subseteq \text{supp}(\sigma)}} \max_{i \in \mathcal{I}} \log F_{r,z}^{(i)}(\sigma) = \inf_{\substack{\sigma \in \mathcal{S}(\mathcal{X}) \\ \text{supp}(\sigma) = \text{supp}(P)}} \max_{i \in \mathcal{I}} \log F_{r,z}^{(i)}(\sigma). \quad (22)$$

Proof. We first consider $r \neq 0$. For $r > 0$, we may assume $P \not\perp \sigma$, since $P \perp \sigma$ gives $F_{r,z}^{(i)}(\sigma) = 0$ for every i and does not affect either supremum. For $r < 0$, we take $\sigma \in \mathcal{S}(\mathcal{X})$ satisfying $\text{supp}(P) \subseteq \text{supp}(\sigma)$. In either case, define

$$\hat{\sigma} = (P\sigma^r P)^{1/r}, \quad c = \text{Tr } \hat{\sigma}, \quad \tilde{\sigma} = \frac{\hat{\sigma}}{c}. \quad (23)$$

Proposition 2.4 gives $0 < c \leq 1$. The state $\tilde{\sigma}$ has support contained in $\text{supp}(P)$ for $r > 0$, and equal to $\text{supp}(P)$ for $r < 0$. Thus Lemma 2.1 and homogeneity give simultaneously

$$F_{r,z}^{(i)}(\tilde{\sigma}) = c^{-rz} F_{r,z}^{(i)}(\sigma), \quad i \in \mathcal{I}. \quad (24)$$

The state $\tilde{\sigma}$ and the number c are the same for every i . Taking logarithms in (24) gives

$$\log F_{r,z}^{(i)}(\tilde{\sigma}) = \log F_{r,z}^{(i)}(\sigma) - rz \log c, \quad i \in \mathcal{I}. \quad (25)$$

Since the additive term $-rz \log c$ is independent of i , we obtain

$$\sum_{i \in \mathcal{I}} p_i \log F_{r,z}^{(i)}(\tilde{\sigma}) = \sum_{i \in \mathcal{I}} p_i \log F_{r,z}^{(i)}(\sigma) - rz \log c \sum_{i \in \mathcal{I}} p_i, \quad (26)$$

$$\min_{i \in \mathcal{I}} \log F_{r,z}^{(i)}(\tilde{\sigma}) = \min_{i \in \mathcal{I}} \log F_{r,z}^{(i)}(\sigma) - rz \log c, \quad (27)$$

$$\max_{i \in \mathcal{I}} \log F_{r,z}^{(i)}(\tilde{\sigma}) = \max_{i \in \mathcal{I}} \log F_{r,z}^{(i)}(\sigma) - rz \log c. \quad (28)$$

If $r > 0$, then $-rz \log c \geq 0$. Hence replacing σ by $\tilde{\sigma}$ cannot decrease the positively weighted sum or the minimum. Therefore, for each of these two objectives, the unrestricted supremum is at most the restricted supremum. The reverse inequality follows because the restricted domain is contained in $\mathcal{S}(\mathcal{X})$. This proves both identities for $r > 0$.

If $r < 0$, then $-rz \log c \leq 0$, so the positively weighted sum and the maximum cannot increase under the same replacement. Every state satisfying $\text{supp}(P) \subseteq \text{supp}(\sigma)$ therefore admits a state with support exactly $\text{supp}(P)$ and an objective value no larger. Consequently, for each of these

two objectives, the restricted infimum is at most the original infimum. Conversely, the original infimum is at most the restricted infimum because states with support exactly $\text{supp}(P)$ form a subset of the original domain. This proves both identities for $r < 0$.

It remains to consider $r = 0$. Let

$$\tilde{\sigma} = \frac{P}{\text{Tr}(P)}. \quad (29)$$

For every $\sigma \in \mathcal{S}(\mathcal{X})$, Lemma 2.2 and the compression identity give simultaneously

$$F_{0,z}^{(i)}(\sigma) \leq \text{Tr}(\Phi_i(I_{\mathcal{X}}))^z = \text{Tr}(\Phi_i(P))^z = F_{0,z}^{(i)}(\tilde{\sigma}), \quad i \in \mathcal{I}. \quad (30)$$

Taking logarithms preserves these inequalities. Thus, the positively weighted sum, the minimum, and the maximum are all maximized by the same state $\tilde{\sigma}$. Since $\tilde{\sigma}$ has support exactly $\text{supp}(P)$, it belongs to each restricted domain. The restricted and unrestricted suprema therefore coincide, proving all three identities for $r = 0$. $\square$

Many of the applications below involve only one positive map. For this case, the theorem reduces to support reduction for $\log F_{r,z}$, as stated in the following corollary.

Corollary 2.6 (Single-map support reduction). *Assume that the relevant $P \neq 0$. The following support reductions hold:*

$$\sup_{\sigma \in \mathcal{S}(\mathcal{X})} \log F_{r,z}(\sigma) = \sup_{\substack{\sigma \in \mathcal{S}(\mathcal{X}) \\ \text{supp}(\sigma) \subseteq \text{supp}(P)}} \log F_{r,z}(\sigma), \quad r > 0, \quad (31)$$

$$\sup_{\sigma \in \mathcal{S}(\mathcal{X})} \log F_{0,z}(\sigma) = \sup_{\substack{\sigma \in \mathcal{S}(\mathcal{X}) \\ \text{supp}(\sigma) = \text{supp}(P)}} \log F_{0,z}(\sigma), \quad (32)$$

$$\inf_{\substack{\sigma \in \mathcal{S}(\mathcal{X}) \\ \text{supp}(P) \subseteq \text{supp}(\sigma)}} \log F_{r,z}(\sigma) = \inf_{\substack{\sigma \in \mathcal{S}(\mathcal{X}) \\ \text{supp}(\sigma) = \text{supp}(P)}} \log F_{r,z}(\sigma), \quad r < 0. \quad (33)$$

Proof. Taking $\mathcal{I} = \{1\}$, $\Phi_1 = \Phi$, and $p_1 = 1$ in Theorem 2.5 gives the logarithmic identities and the stated maximizing state for $r = 0$. $\square$

3. APPLICATIONS TO QUANTUM INFORMATION MEASURES

The examples below are applications of Theorem 2.5 and its single-map specialization in Corollary 2.6, obtained by identifying the relevant projection and positive maps in each example. Throughout this section, $\alpha \in (0, 1) \cup (1, \infty)$ and $z > 0$.

Example 3.1 (Three-parameter conditional entropy). *Let $\rho_{AB} \in \mathcal{S}(A \otimes B)$, and consider the three-parameter conditional entropy $H_{\alpha,z}^{\lambda}(A|B)_{\rho}$. Set*

$$P = \Pi_{\rho_B}, \quad r = \frac{\lambda(1-\alpha)}{z}, \quad (34)$$

and define the positive map $\Phi \in \mathcal{T}(B, A \otimes B)$ by

$$\Phi(X) = \rho_{AB}^{\frac{\alpha}{2z}} \left(I_A \otimes \rho_B^{\frac{(1-\lambda)(1-\alpha)}{2z}} X \rho_B^{\frac{(1-\lambda)(1-\alpha)}{2z}} \right) \rho_{AB}^{\frac{\alpha}{2z}}. \quad (35)$$

This map satisfies $\Phi(X) = \Phi(PXP)$ for every $X \in \text{Pos}(B)$. The variational expression for the entropy can be written as

$$H_{\alpha,z}^\lambda(A|B)_\rho = \frac{1}{1-\alpha} \text{opt}_{\sigma_B} \log F_{r,z}(\sigma_B), \quad (36)$$

where opt is the supremum over $S(B)$ for $r \geq 0$ and the infimum over $\sigma_B \in S(B)$ satisfying $\text{supp}(\rho_B) \subseteq \text{supp}(\sigma_B)$ for $r < 0$. Thus, Corollary 2.6 yields the following identities for the entropy itself:

$$r > 0 : \quad H_{\alpha,z}^\lambda(A|B)_\rho = \frac{1}{1-\alpha} \sup_{\substack{\sigma_B \in S(B) \\ \text{supp}(\sigma_B) \subseteq \text{supp}(\rho_B)}} \log F_{r,z}(\sigma_B), \quad (37)$$

$$r = 0 : \quad H_{\alpha,z}^\lambda(A|B)_\rho = \frac{1}{1-\alpha} \log F_{0,z} \left(\frac{P}{\text{Tr}(P)} \right), \quad (38)$$

$$r < 0 : \quad H_{\alpha,z}^\lambda(A|B)_\rho = \frac{1}{1-\alpha} \inf_{\substack{\sigma_B \in S(B) \\ \text{supp}(\sigma_B) = \text{supp}(\rho_B)}} \log F_{r,z}(\sigma_B). \quad (39)$$

The factor $1/(1-\alpha)$ remains outside the optimization, so these expressions also cover the cases in which passing from the trace functional to the entropy reverses the optimization direction.

Example 3.2 (α - z optimized conditional entropy). Let $\rho_{AB} \in S(A \otimes B)$. We use the α - z Rényi relative entropy $D_{\alpha,z}$ [1]. The optimized α - z conditional entropy is

$$H_{\alpha,z}^\uparrow(A|B)_\rho = \sup_{\sigma_B \in S(B)} -D_{\alpha,z}(\rho_{AB} \| I_A \otimes \sigma_B). \quad (40)$$

Set

$$P = \Pi_{\rho_B}, \quad r = \frac{1-\alpha}{z}, \quad (41)$$

and define the positive map $\Phi \in \mathcal{T}(B, A \otimes B)$ by

$$\Phi(X) = \rho_{AB}^{\frac{\alpha}{2z}} (I_A \otimes X) \rho_{AB}^{\frac{\alpha}{2z}}. \quad (42)$$

This map satisfies $\Phi(X) = \Phi(PXP)$, and, whenever the relative entropy is finite,

$$D_{\alpha,z}(\rho_{AB} \| I_A \otimes \sigma_B) = \frac{1}{\alpha-1} \log F_{r,z}(\sigma_B). \quad (43)$$

For $\alpha \in (0, 1)$, we have $r > 0$. Corollary 2.6 therefore gives

$$H_{\alpha,z}^\uparrow(A|B)_\rho = \sup_{\substack{\sigma_B \in S(B) \\ \text{supp}(\sigma_B) \subseteq \text{supp}(\rho_B)}} -D_{\alpha,z}(\rho_{AB} \| I_A \otimes \sigma_B). \quad (44)$$

For $\alpha \in (1, \infty)$, finite relative entropy requires $\text{supp}(\rho_B) \subseteq \text{supp}(\sigma_B)$. Since $r < 0$, the corollary yields

$$H_{\alpha,z}^\uparrow(A|B)_\rho = \sup_{\substack{\sigma_B \in S(B) \\ \text{supp}(\sigma_B) = \text{supp}(\rho_B)}} -D_{\alpha,z}(\rho_{AB} \| I_A \otimes \sigma_B). \quad (45)$$

In the following three examples, we use $D_{\alpha,z}$ to extend the definitions of doubly minimized Rényi mutual information, Augustin information, and the Rényi information radius to the α - z setting.

Example 3.3 (Doubly minimized Rényi mutual information). *Let $\rho_{AB} \in \mathcal{S}(A \otimes B)$, and consider*

$$I_{\alpha,z}^{\downarrow\downarrow}(A : B)_\rho = \inf_{\substack{\sigma_A \in \mathcal{S}(A) \\ \sigma_B \in \mathcal{S}(B)}} D_{\alpha,z}(\rho_{AB} \| \sigma_A \otimes \sigma_B). \quad (46)$$

Set $r = (1 - \alpha)/z$. For fixed σ_A , define the positive map $\Phi_b \in \mathcal{T}(B, A \otimes B)$ by

$$\Phi_b(X) = \rho_{AB}^{\frac{\alpha}{2z}}(\sigma_A^r \otimes X)\rho_{AB}^{\frac{\alpha}{2z}}. \quad (47)$$

Whenever the relative entropy is finite, we have

$$D_{\alpha,z}(\rho_{AB} \| \sigma_A \otimes \sigma_B) = \frac{1}{\alpha - 1} \log \text{Tr}(\Phi_b(\sigma_B^r))^z. \quad (48)$$

This map satisfies the compression identity with $P = \Pi_{\rho_B}$. Corollary 2.6 allows σ_B to be replaced by a state supported on $\text{supp}(\rho_B)$ without increasing the relative entropy. Fixing this new state, define the positive map $\Phi_a \in \mathcal{T}(A, A \otimes B)$ by

$$\Phi_a(X) = \rho_{AB}^{\frac{\alpha}{2z}}(X \otimes \sigma_B^r)\rho_{AB}^{\frac{\alpha}{2z}}. \quad (49)$$

On the same finite-relative-entropy domain,

$$D_{\alpha,z}(\rho_{AB} \| \sigma_A \otimes \sigma_B) = \frac{1}{\alpha - 1} \log \text{Tr}(\Phi_a(\sigma_A^r))^z. \quad (50)$$

This map satisfies the compression identity with $P = \Pi_{\rho_A}$, so the same corollary reduces the support of σ_A . For $\alpha > 1$, finite relative entropy initially requires both marginal supports to be contained in the corresponding reference-state supports. The resulting support reductions are as follows. For $\alpha \in (0, 1)$,

$$I_{\alpha,z}^{\downarrow\downarrow}(A : B)_\rho = \inf_{\substack{\sigma_A \in \mathcal{S}(A), \sigma_B \in \mathcal{S}(B) \\ \text{supp}(\sigma_A) \subseteq \text{supp}(\rho_A) \\ \text{supp}(\sigma_B) \subseteq \text{supp}(\rho_B)}} D_{\alpha,z}(\rho_{AB} \| \sigma_A \otimes \sigma_B). \quad (51)$$

For $\alpha \in (1, \infty)$,

$$I_{\alpha,z}^{\downarrow\downarrow}(A : B)_\rho = \inf_{\substack{\sigma_A \in \mathcal{S}(A), \sigma_B \in \mathcal{S}(B) \\ \text{supp}(\sigma_A) = \text{supp}(\rho_A) \\ \text{supp}(\sigma_B) = \text{supp}(\rho_B)}} D_{\alpha,z}(\rho_{AB} \| \sigma_A \otimes \sigma_B). \quad (52)$$

Example 3.4 (Augustin information). *Let $\{p_x, \rho_x\}_{x \in \mathcal{X}}$ be a finite ensemble of states on $\mathcal{H}$, with $p_x > 0$ for every x and $\sum_x p_x = 1$. Consider the quantity*

$$I_{\alpha,z}^a(p, \{\rho_x\}) = \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \sum_x p_x D_{\alpha,z}(\rho_x \| \sigma). \quad (53)$$

Set $P = \Pi_{\sum_x \rho_x}$ and $r = (1 - \alpha)/z$, and use the positive maps $\Phi_x \in \mathcal{T}(\mathcal{H}, \mathcal{H})$ given by

$$\Phi_x(X) = \rho_x^{\frac{\alpha}{2z}} X \rho_x^{\frac{\alpha}{2z}}. \quad (54)$$

Each map satisfies $\Phi_x(X) = \Phi_x(PXP)$. On the finite-objective domain,

$$\sum_x p_x D_{\alpha,z}(\rho_x \| \sigma) = \frac{1}{\alpha - 1} \sum_x p_x \log F_{r,z}^{(x)}(\sigma). \quad (55)$$

For $\alpha < 1$, minimizing the left-hand side amounts to maximizing the weighted logarithmic sum; for $\alpha > 1$, it amounts to minimizing that sum. So Theorem 2.5 yields the following identities. For $\alpha \in (0, 1)$,

$$I_{\alpha,z}^a(p, \{\rho_x\}) = \inf_{\substack{\sigma \in \mathcal{S}(\mathcal{H}) \\ \text{supp}(\sigma) \subseteq \text{supp}(P)}} \sum_x p_x D_{\alpha,z}(\rho_x \| \sigma). \quad (56)$$

For $\alpha \in (1, \infty)$,

$$I_{\alpha,z}^a(p, \{\rho_x\}) = \inf_{\substack{\sigma \in \mathcal{S}(\mathcal{H}) \\ \text{supp}(\sigma) = \text{supp}(P)}} \sum_x p_x D_{\alpha,z}(\rho_x \| \sigma). \quad (57)$$

Example 3.5 (Rényi information radius). Let $\{\rho_x\}_{x \in \mathcal{X}}$ be a nonempty finite family of states on $\mathcal{H}$. Its Rényi information radius is

$$R_{\alpha,z}(\{\rho_x\}) = \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \max_{x \in \mathcal{X}} D_{\alpha,z}(\rho_x \| \sigma). \quad (58)$$

Set $P = \Pi_{\sum_x \rho_x}$ and $r = (1 - \alpha)/z$, and use the positive maps from Example 3.4. For reference states for which all the relative entropies are finite,

$$\alpha \in (0, 1) : \quad \max_{x \in \mathcal{X}} D_{\alpha,z}(\rho_x \| \sigma) = \frac{1}{\alpha - 1} \min_{x \in \mathcal{X}} \log F_{r,z}^{(x)}(\sigma), \quad (59)$$

$$\alpha \in (1, \infty) : \quad \max_{x \in \mathcal{X}} D_{\alpha,z}(\rho_x \| \sigma) = \frac{1}{\alpha - 1} \max_{x \in \mathcal{X}} \log F_{r,z}^{(x)}(\sigma). \quad (60)$$

The minimum and maximum identities in Theorem 2.5 therefore apply. For $\alpha \in (0, 1)$,

$$R_{\alpha,z}(\{\rho_x\}) = \inf_{\substack{\sigma \in \mathcal{S}(\mathcal{H}) \\ \text{supp}(\sigma) \subseteq \text{supp}(P)}} \max_{x \in \mathcal{X}} D_{\alpha,z}(\rho_x \| \sigma). \quad (61)$$

For $\alpha \in (1, \infty)$,

$$R_{\alpha,z}(\{\rho_x\}) = \inf_{\substack{\sigma \in \mathcal{S}(\mathcal{H}) \\ \text{supp}(\sigma) = \text{supp}(P)}} \max_{x \in \mathcal{X}} D_{\alpha,z}(\rho_x \| \sigma). \quad (62)$$

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